Spurious solitons and structural stability of finite difference schemes for nonlinear wave equations

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Abstract

The goal of this work is to determine classes of traveling solitary wave solutions for a differential approximation of a finite difference scheme by means of a hyperbolic ansatz.

1 Introduction

The Burgers equation:

$$u_t + c\,u\,u_x - \mu\,u_{xx} = 0, \tag{1}$$

α, μ being real constants, plays a crucial role in the history of wave equations. It was named after its use by Burgers [1] for studying turbulence in 1939.

A finite difference scheme for the Burgers equation can be written under the following form:

$$F({u_l}^m, h, \tau) = 0, \tag{2}$$

where:

$${u_l}^m = u\,(l\,dx, m\,dt) \tag{3}$$

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$l \in \{i-1,\, i,\, i+1\}$, $m \in \{n-1,\, n,\, n+1\}$, $j = 1, \ldots, n_x$, $n = 1, \ldots, n_t$, h, τ denoting respectively the mesh size and time step, and σ the Courant-Friedrichs-Lewy number (cfl) coefficient, defined as $\sigma = c\,\tau/h$.

A numerical scheme is specified by selecting appropriate expression of the function F in equation (2).

Considering the ${u_l}^m$ terms as functions of the mesh size h and time step τ, expanding them at a given order by means of their Taylor series, and neglecting the $o(\tau^p)$ and $o(h^q)$ terms, for given values of the integers p, q, lead to a differential approximation of the Burgers equation, of the form:

$$\mathcal{F}(u, \frac{\partial^r u}{\partial x^r}, \frac{\partial^s u}{\partial t^s}, h, \tau) = 0, \tag{4}$$

r, s being integers.

For sake of simplicity, a non-dimensional form of (4) will be used:

$$\widetilde{\mathcal{F}}(\tilde{u}, \frac{\partial^r \tilde{u}}{\partial \tilde{x}^r}, \frac{\partial^s \tilde{u}}{\partial \tilde{t}^s}) = 0, \tag{5}$$

Depending on this differential approximation (4), solutions, as solitary waves, may arise.

The paper is organized as follows. Two specific schemes are exhibited in section 2. The general method is exposed in Section 3. In Section 4, it is shown that out of the two studied schemes, only one leads to solitary waves. A related class of traveling wave solutions of equation (4) is thus presented, by using a hyperbolic ansatz. The stability of this class of solutions is discussed in section 5. A numerical example is exposed in section 6.

2 Scheme study

2.1 Finite second-order centered scheme in space, Euler-time scheme

For the finite second-order centered scheme in space and Euler-time scheme, the function F of (2) takes the form:

$$F({u_l}^m, h, \tau) = \frac{{u_i}^{n+1} - {u_i}^n}{\tau} + c\, {u_i}^n\, \frac{{u_{i+1}}^n - {u_{i-1}}^n}{2\,h} - \mu\, \frac{{u_{i+1}}^n - 2\,{u_i}^n + {u_{i-1}}^n}{h^2} = 0 \tag{6}$$

Consider ${u_i}^{n+1}$ as a function of the time step τ, and expand it at the second order by means of its Taylor series:

$$ {u_i}^{n+1} = u\,(i\,h, (n+1)\,\tau) = u\,(i\,h, n\,\tau) + \tau\,u_t\,(i\,h, n\,\tau) + \frac{\tau^2}{2}\,u_{tt}\,(i\,h, n\,\tau) + o(\tau^2) \quad (7) $$

It ensures:

$$ \frac{{u_i}^{n+1} - {u_i}^n}{\tau} = u_t\,(i\,h, n\,\tau) + \frac{\tau}{2}\,u_{tt}\,(i\,h, n\,\tau) + o(\tau) \quad (8) $$

In the same way, consider ${u_{i+1}}^n$ and ${u_{i-1}}^n$ as functions of the mesh size h, and expand them at the fourth order by means of their Taylor series:

$$ \begin{aligned} {u_{i+1}}^n =& u\,((i+1)\,h, n\,\tau) \\ =& u\,(i\,h, n\,\tau) + h\,u_x\,(i\,h, n\,\tau) + \frac{h^2}{2}\,u_{xx}\,(i\,h, n\,\tau) + \frac{h^3}{3!}\,u_{xxx}\,(i\,h, n\,\tau) + \frac{h^4}{4!}\,u_{xxxx}\,(i\,h, n\,\tau) + o(h^4) \end{aligned} \quad (9) $$

$$ \begin{aligned} {u_{i-1}}^n =& u\,((i-1)\,h, n\,\tau) \\ =& u\,(i\,h, n\,\tau) - h\,u_x\,(i\,h, n\,\tau) + \frac{h^2}{2}\,u_{xx}\,(i\,h, n\,\tau) - \frac{h^3}{3!}\,u_{xxx}\,(i\,h, n\,\tau) + \frac{h^4}{4!}\,u_{xxxx}\,(i\,h, n\,\tau) + o(h^4) \end{aligned} \quad (10) $$

It ensures:

$$ \frac{{u_{i+1}}^n - 2\,{u_i}^n + {u_{i-1}}^n}{h^2} = u_{xx}\,(i\,h, n\,\tau) + \frac{2\,h^2}{4!}\,u_{xxxx}\,(i\,h, n\,\tau) + o(h^2) \quad (11) $$

and:

$$ \frac{{u_{i+1}}^n - {u_{i-1}}^n}{2\,h} = u_x\,(i\,h, n\,\tau) + \frac{h^2}{3!}\,u_{xxx}\,(i\,h, n\,\tau) + o(h^3) \quad (12) $$

Equation (4) can thus be written as:

$$ \begin{aligned} & u_t\,(i\,h, n\,\tau) + \frac{\tau}{2}\,u_{tt}\,(i\,h, n\,\tau) + o(\tau) \\ & + c\,u\,(i\,h, n\,\tau)\left[u_x\,(i\,h, n\,\tau) + \frac{h^2}{3!}\,u_{xxx}\,(i\,h, n\,\tau) + o(h^3)\right] \\ & - \mu\left[u_{xx}\,(i\,h, n\,\tau) + \frac{2\,h^2}{4}\,u_{xxxx}\,(i\,h, n\,\tau) + o(h^2)\right] = 0 \end{aligned} \quad (13) $$

i. e., at $x = i\,h$ and $t = n\,\tau$:

$$ \left[u_t + \frac{\tau}{2}\,u_{tt} + o(\tau) + c\,u\left[u_x + \frac{h^2}{3!}\,u_{xxx} + o(h^3)\right] - \mu\left[u_{xx} + \frac{2\,h^2}{4!}\,u_{xxxx} + o(h^2)\right]\right]_{(x,t)} = 0 \quad (14) $$

The first differential approximation of the Burgers equation (1) is thus obtained neglecting the $o(\tau)$ and $o(h^2)$ terms:

$$\left[u_t + \frac{\tau}{2}\, u_{tt} + c\, u \left[u_x + \frac{h^2}{3!}\, u_{xxx} \right] - \mu \left[u_{xx} + \frac{h^2}{12}\, u_{xxxx} \right] \right]_{(x,t)} = 0 \tag{15}$$

that we will keep as:

$$u_t + c\, u\, u_x - \mu\, u_{xx} + \frac{\tau}{2}\, u_{tt} + \frac{h^2}{6}\, u\, u_{xxx} - \mu\, \frac{h^2}{12}\, u_{xxxx} = 0 \tag{16}$$

For sake of simplicity, this latter equation can be adimensionalized through in the following way:
set:

$$\left\{ \begin{array}{rcl} u & = & U_0\, \tilde{u} \\ t & = & \tau_0\, \tilde{t} \\ x & = & h_0\, \tilde{x} \end{array} \right. \tag{17}$$

where:

$$U_0 = \frac{h_0}{\tau_0} \tag{18}$$

Denote by Re_h the mesh Reynolds number, defined as:

$$Re_h = \frac{U_0\, h}{\mu} \tag{19}$$

The change of variables (17) leads to:

$$\left\{ \begin{array}{rcl} u_t & = & \frac{U_0}{\tau_0}\, \tilde{u}_{\tilde{t}} \\ u_{x^k} & = & \frac{U_0}{h_0^k}\, \tilde{u}_{\tilde{x}^k} \end{array} \right. \tag{20}$$

Multiplying (16) by $\frac{\tau_0}{U_0}$ yields:

$$\tilde{u}_{\tilde{t}} + c\, \frac{U_0\, \tau_0}{h_0}\, \tilde{u}\, \tilde{u}_{\tilde{x}} - \mu\, \frac{\tau_0}{h_0^2}\, \tilde{u}_{\tilde{x}\tilde{x}} + \frac{\tau}{2\, \tau_0}\, \tilde{u}_{\tilde{t}\tilde{t}} + \frac{h^2\, U_0 \tau_0}{6\, h_0^3}\, \tilde{u}\, \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} - \mu\, \frac{h^2\, \tau_0}{12\, h_0^4}\, \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = 0 \tag{21}$$

Relations (18) and (19) ensure:

$$\tilde{u}_{\tilde{t}} + c\, \tilde{u}\, \tilde{u}_{\tilde{x}} - \frac{h}{h_0\, Re_h}\, \tilde{u}_{\tilde{x}\tilde{x}} + \frac{\tau}{2\, \tau_0}\, \tilde{u}_{\tilde{t}\tilde{t}} + \frac{h^2}{6\, h_0^2}\, \tilde{u}\, \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} - \frac{h^3}{12\, Re_h\, h_0^3}\, \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = 0 \tag{22}$$

For $h = h_0$, due to $\sigma = \frac{U_0\, \tau}{h}$, (23) becomes:

$$\tilde{u}_{\tilde{t}} + c\, \tilde{u}\, \tilde{u}_{\tilde{x}} - \frac{1}{Re_h}\, \tilde{u}_{\tilde{x}\tilde{x}} + \sigma\, \tilde{u}_{\tilde{t}\tilde{t}} + \frac{1}{6}\, \tilde{u}\, \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} - \frac{1}{12\, Re_h}\, \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = 0 \tag{23}$$

2.2 The Lax-Wendroff scheme

For the Lax-Wendroff scheme, the function F of (2) takes the form:

$$F(u_l{}^m, h, \tau) = \frac{u_i{}^{n+1} - u_i{}^n}{\tau} + c\, u_i{}^n \left\{ \frac{u_{i+1}{}^n - u_{i-1}{}^n}{2\, h} \right\} - (\mu + \frac{c^2\, \tau}{2}) \left\{ \frac{u_{i+1}{}^n - 2\, u_i{}^n + u_{i-1}{}^n}{h^2} \right\} = 0 \tag{24}$$

$\frac{u_i{}^{n+1} - u_i{}^n}{\tau}$ is expressed by means of (8), and $\frac{u_{i+1}{}^n - 2\, u_i{}^n + u_{i-1}{}^n}{h^2}$ by means of (11):

$$\frac{u_{i+1}{}^n - 2\, u_i{}^n + u_{i-1}{}^n}{h^2} = u_{xx}\,(i\, h, n\, \tau) + \frac{2\, h^2}{4!}\, u_{xxxx}\,(i\, h, n\, \tau) + o(h^2) \tag{25}$$

(11) also yields:

$$\frac{u_{i+1}{}^n - u_{i-1}{}^n}{2\, h} = u_x\,(i\, h, n\, \tau) + \frac{h^2}{3!}\, u_{xxx}\,(i\, h, n\, \tau) + o(h^3) \tag{26}$$

Equation (24) can thus be written as:

$$\begin{aligned} & u_t\,(i\, h, n\, \tau) + \frac{\tau}{2}\, u_{tt}\,(i\, h, n\, \tau) + o(\tau) \\ & + \alpha\, u\,(i\, h, n\, \tau)\left[u_x\,(i\, h, n\, \tau) + \frac{h^2}{3!}\, u_{xxx}\,(i\, h, n\, \tau) + o(h^3)\right] \\ & - (\mu + \frac{c^2\, \tau}{2})\left[u_{xx}\,(i\, h, n\, \tau) + \frac{2\, h^2}{4!}\, u_{xxxx}\,(i\, h, n\, \tau) + o(h^2)\right] = 0 \end{aligned} \tag{27}$$

i. e., at $x = i\, h$ and $t = n\, \tau$:

$$\left[u_t + \frac{\tau}{2}\, u_{tt} + o(\tau) + c\, u \left[u_x + \frac{h^2}{3!}\, u_{xxx} + o(h^3) \right] - (\mu + \frac{c^2\, \tau}{2}) \left[u_{xx} + \frac{2\, h^2}{4!}\, u_{xxxx} + o(h^2) \right] \right]_{(x,t)} = 0 \tag{28}$$

The first differential approximation of the Burgers equation (1) is thus obtained neglecting the $o(\tau)$ and $o(h^2)$ terms:

$$\left[u_t + \frac{\tau}{2}\, u_{tt} + c\, u \left[u_x + \frac{h^2}{3!}\, u_{xxx} \right] - (\mu + \frac{c^2\, \tau}{2}) \left[u_{xx} + \frac{h^2}{12}\, u_{xxxx} \right] \right]_{(x,t)} = 0 \tag{29}$$

that we will keep as:

$$u_t + c\, u\, u_x - (\mu + \frac{c^2\, \tau}{2\, h^2})\, u_{xx} + \frac{\tau}{2}\, u_{tt} + \frac{h^2}{6}\, u\, u_{xxx} - (\mu + \frac{c^2\, \tau}{2})\, \frac{h^2}{12}\, u_{xxxx} = 0 \tag{30}$$

Equation (30) is adimensionalized as in section 2.1, leading to:

$$\tilde{u}_{\tilde{t}} + c\, \tilde{u}\, \tilde{u}_x - (\frac{1}{Re_h} + \frac{c^2\, \sigma}{2})\, \tilde{u}_{\tilde{x}\tilde{x}} + \frac{\tau}{2}\, \tilde{u}_{\tilde{t}\tilde{t}} + \frac{1}{6}\, \tilde{u}\, \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} - (\frac{1}{Re_h} + \frac{c^2\, \sigma}{2})\, \frac{1}{12}\, \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = 0 \tag{31}$$

3 Solitary waves

Approximated solutions of the Burgers equation (1) by means of the difference scheme (2) strongly depend on the values of the time and space steps. For specific values of τ and h, equation (5) can, for instance, have traveling wave solutions which can be of great disturbance to the searched solution.
We presently aim at determining the conditions, depending on the values of the parameters τ and h, which give birth to traveling wave solutions of (16).
Following Feng [2] and our previous work [3], where traveling wave solutions of the cBKDV equation were exhibited as combinations of bell-profile waves and kink-profile waves, we aim at determining traveling wave solutions of (5).
Following [2], we assume that equation (5) has the traveling wave solution of the form

$$\tilde{u}(\tilde{x},\tilde{t}) = \tilde{u}(\xi), \quad \xi = \tilde{x} - v\,\tilde{t} \tag{32}$$

where v is the wave velocity. Substituting (32) into equation (5) leads to:

$$\widetilde{\mathcal{F}}_\xi(\tilde{u}, \tilde{u}^{(r)}, (-v)^s\,\tilde{u}^{(s)}) = 0, \tag{33}$$

Performing an integration of (33) with respect to ξ and setting the integration constant to zero leads to an equation of the form:

$$\widetilde{\mathcal{F}}_\xi^{\mathcal{P}}(\tilde{u}, \tilde{u}^{(r)}, (-v)^s\,\tilde{u}^{(s)}) = 0, \tag{34}$$

which will be the starting point for the determination of solitary waves solutions.

4 Traveling Solitary Waves

4.1 Hyperbolic Ansatz

The discussion in the preceding section provides us useful information when we construct traveling solitary wave solutions for equation (33). Based on this result, in this section, a class of traveling wave solutions of the equivalent equation (16) are searched as a combination of bell-profile waves and kink-profile waves of the form

$$\tilde{u}(\tilde{x},\tilde{t}) = \sum_{i=1}^{n} \left(U_i \tanh^i \left[C_i(\tilde{x} - v\,\tilde{t}) \right] + V_i \operatorname{sech}^i \left[C_i(\tilde{x} - v\,\tilde{t} + x_0) \right] \right) + V_0 \tag{35}$$

where the $U_i's$, $V_i's$, $C_i's$, $(i = 1, \cdots, n)$, V_0 and v are constants to be determined. In the following, c is taken equal to 1.

4.2 Theoretical study

Substitution of (35) into equation (33) leads to an equation of the form

$$\sum_{i,\,j,\,k} A_i \tanh^i\big(C_i\,\xi\big)\,\mathrm{sech}^j\big(C_i\,\xi\big)\,\sinh^k\big(C_i\,\xi\big) = 0 \tag{36}$$

the A_i being real constants.
The difficulty for solving equation (36) lies in finding the values of the constants U_i, V_i, C_i, V_0 and v by using the over-determined algebraic equations. Following [2], after balancing the higher-order derivative term and the leading nonlinear term, we deduce $n = 1$. Then, following [3] we replace $\mathrm{sech}(C_1\,\xi)$ by $\frac{2}{e^{C_1\,\xi}+e^{-C_1\,\xi}}$, $\sinh(C_1\,\xi)$ by $\frac{e^{C_1\,\xi}-e^{-C_1\,\xi}}{2}$, $\tanh(C_1\,\xi)$ by $\frac{e^{C_1\,\xi}-e^{-C_1\,\xi}}{e^{C_1\,,\xi}+e^{-C_1\,\xi}}$, and multiply both sides by $\left(e^{\xi\,C_1}+e^{-\xi\,C_1}\right)^5 e^{5\,\xi\,C_1}$, so that equation (36) can be rewritten in the following form:

$$\sum_{k=0}^{10} P_k(U_1,\ V_1,\ C_1,\ v,\ V_0)\,e^{k\,C_1\,\xi}\ =\ 0, \tag{37}$$

where the P_k ($k = 0$, ..., 10), are polynomials of U_1, V_1, C_1, V_0 and v.

4.3 Scheme study

4.3.1 The Finite second-order centered scheme in space, Euler-time scheme

Equation (33) is presently given by:

$$-v\,\tilde{u}'(\xi)+c\,\tilde{u}(\xi)\,\tilde{u}'(\xi)-\frac{1}{Re_h}\,\tilde{u}''(\xi)+v^2\,\frac{\tau}{2}\,\tilde{u}''(\xi)+\frac{1}{6}\,\tilde{u}(\xi)\,\tilde{u}^{(3)}(\xi)-\frac{1}{Re_h}\,\frac{1}{12}\,\tilde{u}^{(4)}(\xi)=0 \tag{38}$$

Performing an integration of (38) with respect to ξ and setting the integration constant to zero yields:

$$-v\,\tilde{u}(\xi)+\frac{c}{2}\,\tilde{u}^2(\xi)+(v^2\,\frac{\sigma}{2}-\frac{1}{Re_h})\,\tilde{u}'(\xi)+\frac{1}{6}\left\{\tilde{u}(\xi)\,\tilde{u}^{(2)}(\xi)-\frac{1}{2}\,\tilde{u}'^{2}(\xi)\right\}-\frac{1}{12\,Re_h}\,\tilde{u}^{(3)}(\xi)=0 \tag{39}$$

The related system (37) has consistent solutions, which are given in Tables 1.
For sake of simplicity, we use ε to denote 1 or -1.

Table 1:

	σ	v	U_1	V_1	C_1	V_0
Sets 1, 2	$\frac{484\,Re_h}{729}$	$\varepsilon\,\frac{108}{11\sqrt{11}\,Re_h}$	$\varepsilon\,\frac{108}{5\sqrt{11}\,Re_h}$	0	$-\varepsilon\,\frac{6}{\sqrt{11}}$	$-\varepsilon\,\frac{108}{5\sqrt{11}\,Re_h}$
Set 3	$\frac{5\,Re_h\left(17\,C_1^2-12\right)}{6C_1^2\left(4C_1^2-9\right)^2}$	$-\frac{2\left(4C_1^3-9C_1\right)}{5\,Re_h}$	$-\frac{18C_1}{5\,Re_h}$	0	$\in\mathbb{R}$	$\frac{18C_1}{5\,Re_h}$
Set 4	$-\frac{Re_h\left(64C_1^6-384C_1^4+551C_1^2-156\right)}{6C_1^2\left(4C_1^2-9\right)^2}$	$\frac{-\frac{5C_1}{13-8C_1^2}-C_1}{Re_h}$	$-\frac{2\left(8C_1^3-9C_1\right)}{Re_h\left(8C_1^2-13\right)}$	0	$\in\mathbb{R}$	$\frac{18C_1}{Re_h\left(8C_1^2-13\right)}$

In the following, we shall denote:

$$\left\{\begin{array}{lll}\sigma_{1,2} & = & \frac{484\,Re_h}{729}\\ \sigma_3 & = & \frac{5\,Re_h\left(17\,C_1^2-12\right)}{6C_1^2\left(4C_1^2-9\right)^2}\\ \sigma_4 & = & -\frac{Re_h\left(64C_1^6-384C_1^4+551C_1^2-156\right)}{6C_1^2\left(4C_1^2-9\right)^2} = -\frac{Re_h\,(C_1^2-4)\,(8\,C_1^2-13)\,(8\,C_1^2-3)}{6C_1^2\left(4C_1^2-9\right)^2}\end{array}\right. \tag{40}$$

4.3.2 The Lax-Wendroff scheme

Equation (33) is then given by:

$$-v\,\tilde{u}'(\xi)+c\,\tilde{u}(\xi)\,u'(\xi)-\left(\frac{1}{Re_h}+\frac{c^2\,\sigma}{2}\right)\tilde{u}''(\xi)+v^2\,\frac{\sigma}{2}\,\tilde{u}''(\xi)+\frac{1}{6}\,\tilde{u}(\xi)\,u^{(3)}(\xi)-(\frac{1}{Re_h}+\frac{c^2\,\sigma}{2})\,\frac{1}{12}\,u^{(4)}(\xi)=0 \tag{41}$$

Performing an integration with respect to ξ and setting the integration constant to zero yields:

$$-v\,\tilde{u}(\xi)+\frac{c}{2}\,\tilde{u}^2(\xi)+(v^2\,\frac{\sigma}{2}-(\frac{1}{Re_h}+\frac{c^2\,\sigma}{2}))\,u'(\xi)+\frac{1}{6}\left[\tilde{u}(\xi)\,\tilde{u}^{(2)}(\xi)-\frac{1}{2}\,\tilde{u}'^2(\xi)\right]-\left(\frac{1}{Re_h}+\frac{c^2\,\sigma}{2}\right)\frac{1}{12}\,\tilde{u}^{(3)}(\xi)=0 \tag{42}$$

The related system (37) does not admit consistent solutions.

5 Stability study

In the following, the stability of the solutions presented in section 4.3.1 is discussed.

The variations of the cfl coefficient σ as a function of the parameters Re_h, C_1, have crucial influence on the stability.

σ_3, σ_4 being even functions of C_1, we shall restrain our study to $C_1 \in]\,0, +\infty\,[$. Calculation yield:

$$\begin{cases} \dfrac{\partial \frac{\sigma_3}{Re_h}}{\partial C_1^2} = -\dfrac{20\left(34\,{C_1}^4-36\,{C_1}^2+27\right)}{3\,C_1^3\left(4\,C_1^2-9\right)^3} > 0 \;, & \dfrac{\partial\left[\sigma_3\cdot Re_h\right]}{\partial C_1^2} = -\dfrac{20\left(34\,{C_1}^4-36\,{C_1}^2+27\right)Re_h^2}{3\,C_1^3\left(4\,C_1^2-9\right)^3} > 0 \\ \dfrac{\partial\left[\frac{\sigma_4}{Re_h}\right]}{\partial C_1^2} = -\dfrac{4\left(96\,C_1^6-238\,C_1^4+468\,C_1^2-351\right)Re_h}{3\,C_1^3\left(4\,C_1^2-9\right)^3} \;, & \dfrac{\partial\left[\sigma_4\cdot Re_h\right]}{\partial C_1^2} = -\dfrac{4\left(96\,C_1^6-238\,C_1^4+468\,C_1^2-351\right)Re_h^2}{3\,C_1^3\left(4\,C_1^2-9\right)^3} \end{cases} \tag{43}$$

Denote by ${C_1^0}^2$ the value of ${C_1}^2$ for which $\frac{\partial\left[\frac{\sigma_4}{Re_h}\right]}{\partial C_1}$ and $\frac{\partial\left[\sigma_4\,Re_h\right]}{\partial C_1}$ vanish.

For a given value of the mesh Reynolds number Re_h, we obtain the following interesting variation tables:

C_1^2	0		$\frac{3}{2}$		$+\infty$
$\frac{\partial\left[\frac{\sigma_3}{Re_h}\right]}{\partial C_1^2}$, $\frac{\partial\left[\sigma_3\cdot Re_h\right]}{\partial C_1^2}$		$+$	$\Vert$	$+$	
$\frac{\sigma_3}{Re_h}$, $\sigma_3\cdot Re_h$			$+\infty \Vert +\infty$		
		$\nearrow$		$\searrow$	
	$-\infty$		$\Vert$		0

(44)

C_1^2	0		${C_1}^{0^2}$		$\frac{3}{2}$		$+\infty$
$\frac{\partial\left[\frac{\sigma_4}{Re_h}\right]}{\partial C_1^2}$, $\frac{\partial\left[\sigma_4\cdot Re_h\right]}{\partial C_1^2}$	$+$		0	$-$	$\Vert$	$+$	
$\frac{\sigma_4}{Re_h}$, $\sigma_4\cdot Re_h$	$+\infty$				$+\infty \Vert +\infty$		
		$\searrow$		$\nearrow$		$\searrow$	
			< 0				< 0

(45)

$\frac{\sigma}{Re_h}$, $\sigma\,Re_h$ take all the values between 0 and $+\infty$. Thus, for stable and unstable sets $\left(\sigma\cdot Re_h\right)$, there exists a solitary wave solution of (39).

6 Numerical Example

In the following, we specifically consider the third traveling solitary wave (see (1)) solution of equation (39).
Numerical values of the parameters are: $Re_h = 1.9$, $C_1 = 3$.

6.1 Analytical soliton

The third traveling solitary wave solitary wave (see (1)) solution of equation (39), is given by:

$$\tilde{u}(\tilde{x}, \tilde{t}) = U_1 \tanh\left[C_1 (\tilde{x} - v\,\tilde{t})\right] + V_0 \tag{46}$$

The variations of (46) a function of the non-dimensional space variable $\tilde{x}$ and the non-dimensional time variable $\tilde{t}$ for $Re_h = 1.9$, $C_1 = 3$, $\sigma = \sigma_3 \simeq 0.034$, is displayed in Figure 1.

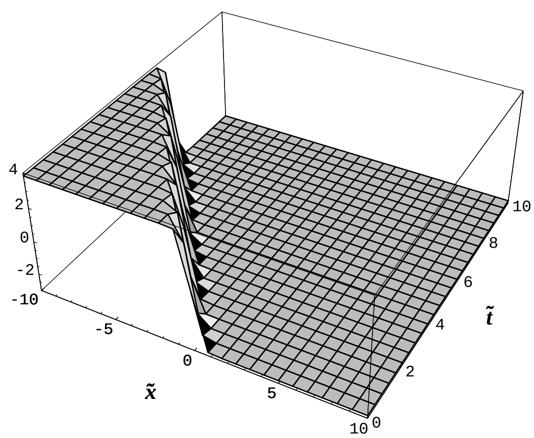

Figure 1: The traveling solitary wave for $Re_h = 1.9$, $C_1 = 3$, $\sigma = \sigma_3 \simeq 0.034$.

Analytical calculation yield:

$$\tilde{u}_{\tilde{t}} + c\,\tilde{u}\,\tilde{u}_{\tilde{x}} - \frac{1}{Re_h}\,\tilde{u}_{\tilde{x}\tilde{x}}\tilde{u}(\tilde{x}, \tilde{t}) = \frac{2\,C_1^2\,(8\,C_1^3 - 9\,C_1)\mathrm{sech}^2\left[C_1\,(\tilde{x} - v\,\tilde{t})\right]}{(8\,C_1^2 - 13)^2\,Re_h^2}\left[8\,C_1^2 - 8 - 18\,Re_h + 2\,(8\,C_1^3 - 9\,C_1)\,\tanh\left[C_1\,(\tilde{x} - v\,\tilde{t})\right]\right] \tag{47}$$

As $\tilde{t}$ increases, $\sup_{\tilde{t},\tilde{x}} \mid \tilde{u}_{\tilde{t}} + c\,\tilde{u}\,\tilde{u}_{\tilde{x}} - \frac{1}{Re_h}\,\tilde{u}_{\tilde{x}\tilde{x}}\tilde{u}(\tilde{x}, \tilde{t}) \mid$ decreases and tends towards 0; hence, the soliton (46) tends towards a solitary wave solution of (1), with an error $o\left(\frac{e^{-2\,(\tilde{x} - v\,\tilde{t})}}{Re_h^2}\right)$.

6.2 Finite difference calculation

Advect n times the soliton (46) through the Finite second-order centered scheme in space, Euler-time scheme 2.1, in the space domain $x \in [x_0, +\infty[$, for:

i. $Re_h = 1.9$, $C_1 = 3$, $\sigma = \sigma_3 \simeq 0.034$, which corresponds to the existence of a solitary wave solution of (4);

ii. $Re_h = 1.9$, $\sigma = 0.06 = \sigma^{'} \neq \sigma_3$, and $Re_h = 1.9$, $\sigma = 0.07 = \sigma^{''} \neq \sigma_3$, which do not correspond to the existence of a solitary wave solution of (4).

The mesh points number is equal to 150, with 100 points in the front wave. x_0 is equal to $-20\,\widetilde{h}$.
Denote:

$$\widetilde{\tau} = \sigma_3 \ , \ \widetilde{\tau}^{'} = \sigma^{'} \ , \ \widetilde{\tau}^{''} = \sigma^{''} \tag{48}$$

The numerical solitary wave, the analytical solitary wave (46), and the numerical solutions at $\widetilde{t} = n_t\,\widetilde{\tau} = E\left[\frac{n_t\,\widetilde{\tau}}{\widetilde{\tau}^{'}}\right]\widetilde{\tau}^{'} = E\left[\frac{n_t\,\widetilde{\tau}}{\widetilde{\tau}^{''}}\right]\widetilde{\tau}^{''}$, where E denotes the entire part, for $n_t = 0$, $n_t = 5$, $n_t = 10$, $n_t = 50$, respectively, are displayed in Fig. 2.

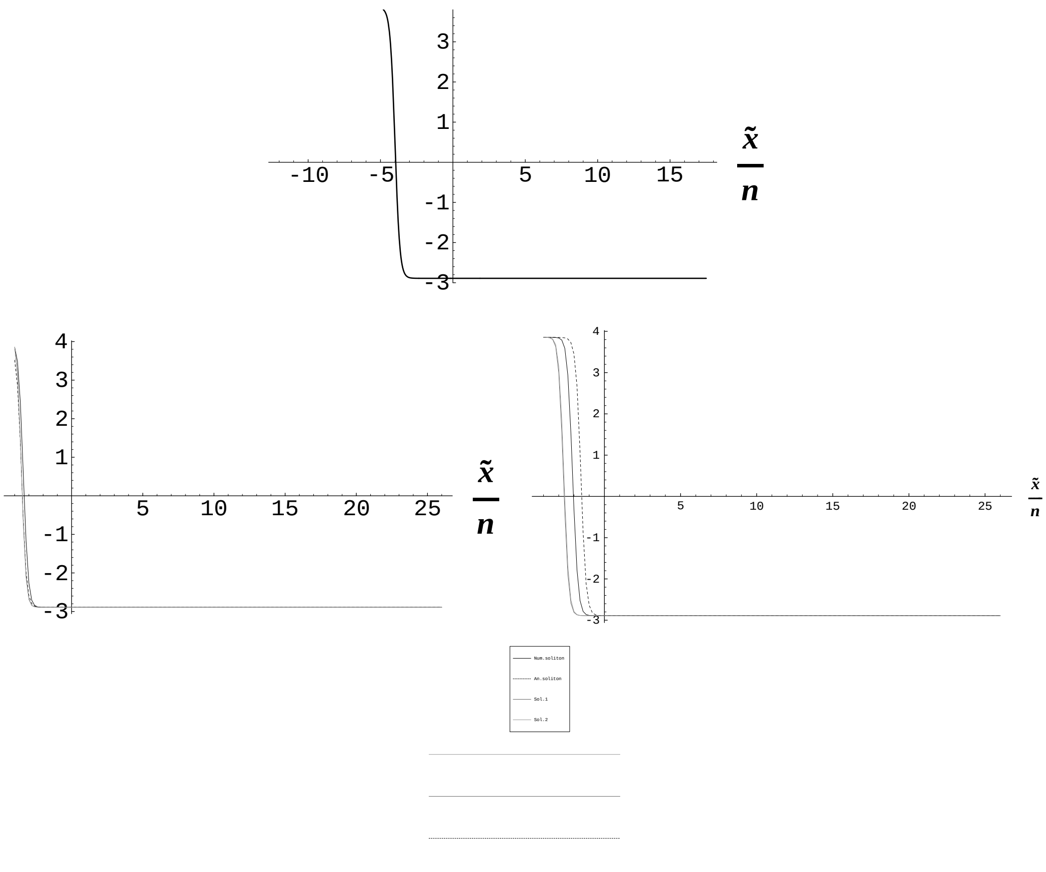

Figure 2: Numerical soliton, analytical soliton, and numerical solutions. Top: $\widetilde{t} = 0$. Bottom: $\widetilde{t} = 10\,\widetilde{\tau}$, $\widetilde{t} = 50\,\widetilde{\tau}$.

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